Revista Integración
Escuela de Matemáticas
Universidad Industrial de Santander
Vol. 22, Nos. 1 y 2, 2004, pág. 67–81

Nonlinear stability for 2 dimensional plane Couette flow

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Abstract. In this expository article, we discuss the application of the resolvent technique to prove nonlinear stability of 2 dimensional plane Couette flow. Using this technique, we show how one can derive a threshold amplitude for perturbations that can lead to turbulence in terms of the parameter called Reynolds number. Our objective is to present this argument in details, trying to be accessible to a wide class of readers, and hopefully catching their attention to the beautiful subject of stability questions in fluid mechanics.

1. Introduction

We discuss nonlinear stability of plane 2 dimensional plane Couette flow via the resolvent method. Applying this method, one can derive lower bounds for the norms of perturbations of the flow that can grow with time. The resolvent method is an interesting technique for proving stability of solutions of differential equations. For a general discussion in various cases, see [3] and [4]. For nonlinear stability of plane Couette flow, this technique has been used in [5], [6]. Linear stability of the flow has been proved in [9].

Application of the resolvent method requires estimates for the resolvent of a linear operator. Moreover, for the specific case of Couette flow, one is interested in determining the exact dependence of the resolvent on a parameter called Reynolds number. This dependence is usually hard to derive. Regarding this question, both numerical and analytical studies can be found in [1], [5], [7], [8], [10]. A complete analytical proof of the dependence of the resolvent on the Reynolds number is still an open problem, as far as we know.

Keywords: Couette flow, resolvent estimates, nonlinear stability.
MSC2000: 76E05, 47A10, 35Q30, 76D05.

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2. The problem

We are interested in the following initial boundary value problem:

$$\begin{cases} u_t + (u\cdot\nabla)u + \nabla p = \dfrac{1}{R}\Delta u, \\ \nabla\cdot u = 0, \\ u(x,0,t) = (0,0), \\ u(x,1,t) = (1,0), \\ u(x,y,t) = u(x+1,y,t), \\ u(x,y,0) = f(x,y), \end{cases} \tag{1}$$

where $u : \mathbb{R}\times[0,1]\times[0,\infty) \longrightarrow \mathbb{R}^2$ is the unknown function $u(x,y,t) = (u_1(x,y,t), u_2(x,y,t))$. The positive parameter R is the Reynolds number. The initial condition $f(x,y)$ is assumed to be divergence free and compatible with the boundary conditions. The pressure $p(x,y,t)$ can be determined in terms of u by the elliptic problem

$$\begin{cases} \Delta p = -\nabla\cdot((u\cdot\nabla)u), \\ p_y(x,0,t) = \dfrac{1}{R}u_{2yy}(x,0,t), \\ p_y(x,1,t) = \dfrac{1}{R}u_{2yy}(x,1,t). \end{cases} \tag{2}$$

It can be easily seen that $U(x,y) = (y,0)$, $P = constant$ is a steady solution of problem (1). The vector field $U(x,y) = (y,0)$ is known as Couette flow. We note that all functions considered here are assumed to be smooth.

Using the resolvent technique, one can prove and quantify asymptotic stability for this flow. By quantification we mean the derivation of a number $M(R)$ such that disturbances of the flow with norm less than $M(R)$ will tend to zero as time t tends to infinity. In other words, deriving a lower bound for the norm of perturbations that can grow with time t.

Our exposition is divided in 3 sections: in section 3, we introduce some basic notation and derive the equations for perturbations of the Couette flow; in section 4 we derive estimates for the solution of the linearized equations for the perturbations; in 5, we use those estimates to prove asymptotic stability for the flow, and to derive the threshold amplitude $M(R)$. In the appendix A, we show the proof of a technical lemma used in Section 5. In the appendix B, the proofs of some simple Sobolev-type inequalities used are given.

3. Notation and equations for the perturbations

We denote by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ the L_2 inner product and norm over $\Omega = [0,1]\times[0,1]$:

$$\langle u,w\rangle = \int_\Omega \overline{u}\cdot w \; dx\,dy\,; \qquad \|u\|^2 = \langle u,u\rangle.$$

All the matrix norms that appear in this paper are the usual Frobenius norms. The usual Sobolev norm of u over Ω is denoted by

$$\|u\|^2_{H^n(\Omega)} = \sum_{j=0}^{n} \|D^j u\|^2,$$

where D^j denotes the j-th derivative of u with respect to the space variables. Unless stated otherwise, all norms in the space variables will be calculated over Ω, and therefore we will write $\|\cdot\|_{H^n(\Omega)}$ as $\|\cdot\|_{H^n}$. We make use of a 2 dimensional version of the weighted norm $\|\cdot\|_{\widetilde{H}}$ used in [5]:

$$\|u\|^2_{\widetilde{H}} = \|u\|^2 + \frac{1}{R}\|Du\|^2 + \frac{1}{R^2}\|u_{xy}\|^2. \tag{3}$$

The maximum norm over Ω is denoted by $|\cdot|_\infty$. The norm $\|\cdot\|_{\widetilde{H}}$ is related with the maximum norm by the Sobolev type inequality (see Appendix B)

$$|\cdot|^2_\infty \le \widetilde{C}R\|\cdot\|^2_{\widetilde{H}}. \tag{4}$$

Since we are interested in functions which are also dependent on time, we use that

$$|u(\cdot,t)|^2_\infty \le \widetilde{C}R\|u(\cdot,t)\|^2_{\widetilde{H}}\ , \quad \forall\, t \ge 0, \tag{5}$$

where $\widetilde{C}$ is a constant independent of any of the parameters.

We are interested in proving asymptotic stability for the Couette flow, which is a stationary solution of (1), that is, to prove that perturbations of the stationary solution that are small enough in some norm will tend to 0 as t tends to infinity. In our presentation here, we show that perturbations having norm $\|\cdot\|_{H^6}$ of order R^{-3} decay with time.

To this aim, let $U = U(x,y)$, $P = P(x,y)$ be a stationary solution of (1). We can obviously use the Couette flow, but we think that the structure of the argument is easier to be understood if one uses any stationary solution. This will not change the estimates we will prove. We derive the equations satisfied by perturbations of this base flow. Let $u(x,y,t)$, $p(x,y,t)$ be a solution of (1) with initial condition $f(x,y) = U(x,y) + \epsilon f'(x,y)$, where f' is divergence free and $\|f'\|_{H^6(\Omega)} = 1$. Then, ϵ defines a unique perturbation amplitude. Write $u(x,y,t) = U(x,y) + \epsilon u'(x,y,t)$ and $p(x,y,t) = P(x,y) + \epsilon p'_1(x,y,t) + \epsilon^2 p'_2(x,y,t)$. Then u',p'_1,p'_2 satisfy the system

$$\begin{cases} u'_t + (u'\cdot\nabla)U + (U\cdot\nabla)u' + \nabla p'_1 + \epsilon(u'\cdot\nabla)u' + \epsilon\nabla p'_2 = \dfrac{1}{R}\Delta u', \\ \nabla\cdot u' = 0, \\ u'(x,0,t) = (0,0), \\ u'(x,1,t) = (0,0), \\ u'(x,y,t) = u'(x+1,y,t), \\ u'(x,y,0) = f'(x,y). \end{cases}$$

The functions p_1' and p_2' are given in terms of u' by

$$\begin{cases} \Delta p_1' = -\nabla\cdot((u'\cdot\nabla)U) - \nabla\cdot((U\cdot\nabla)u'), \\ p_{1y}'(x,0,t) = \dfrac{1}{R}u_{2yy}'(x,0,t), \\ p_{1y}'(x,1,t) = \dfrac{1}{R}u_{2yy}'(x,1,t), \end{cases}$$

and

$$\begin{cases} \Delta p_2' = -\nabla\cdot((u'\cdot\nabla)u'), \\ p_{2y}'(x,0,t) = 0, \\ p_{2y}'(x,1,t) = 0. \end{cases}$$

The functions p_1' and p_2' can be estimated in terms of u' by

$$\begin{aligned} \|\nabla p_1'(\cdot,\cdot,t)\|^2 &\leq C\|u'(\cdot,\cdot,t)\|_{H^3}^2, \qquad \forall\, t\geq 0, \\ \|\nabla p_2'(\cdot,\cdot,t)\|^2 &\leq \|(u'\cdot\nabla)u'(\cdot,\cdot,t)\|^2, \qquad \forall\, t\geq 0. \end{aligned}$$

From now on, to simplify the notation, we drop the $'$ in the equations above, and just write u, p_1, p_2. With this notation, the equations above are

$$\begin{cases} u_t + (u\cdot\nabla)U + (U\cdot\nabla)u + \nabla p_1 + \epsilon(u\cdot\nabla)u + \epsilon\nabla p_2 = \dfrac{1}{R}\Delta u, \\ \nabla\cdot u = 0, \\ u(x,0,t) = (0,0), \\ u(x,1,t) = (0,0), \\ u(x,y,t) = u(x+1,y,t), \\ u(x,y,0) = f(x,y), \end{cases} \tag{6}$$

$$\begin{cases} \Delta p_1 = -\nabla\cdot((u\cdot\nabla)U) - \nabla\cdot((U\cdot\nabla)u), \\ p_{1y}(x,0,t) = \dfrac{1}{R}u_{2yy}(x,0,t), \\ p_{1y}(x,1,t) = \dfrac{1}{R}u_{2yy}(x,1,t), \end{cases} \tag{7}$$

and

$$\begin{cases} \Delta p_2 = -\nabla\cdot((u\cdot\nabla)u), \\ p_{2y}(x,0,t) = 0, \quad p_{2y}(x,1,t) = 0. \end{cases} \tag{8}$$

Note that p_1 depends linearly on u. Moreover, for all $t\geq 0$, we have

$$\|\nabla p_1(\cdot,\cdot,t)\|^2 \leq C\|u(\cdot,\cdot,t)\|_{H^3}^2, \tag{9}$$

$$\|\nabla p_2(\cdot,\cdot,t)\|^2 \leq \|(u\cdot\nabla)u(\cdot,\cdot,t)\|^2. \tag{10}$$

When the initial data are divergence free and the terms of pressure are given by the equations (7) and (8) above, the solution u of problem (6) remains divergence free for all time t. Therefore, we drop the continuity equation and write problem (6) as

$$\begin{cases} u_t = \mathcal{L}u - \epsilon(u\cdot\nabla)u - \epsilon\nabla p_2, \\ u(x,0,t) = (0,0), \\ u(x,1,t) = (0,0), \\ u(x,y,t) = u(x+1,y,t), \\ u(x,y,0) = f(x,y), \end{cases} \tag{11}$$

where $\mathcal{L}$ is a linear operator depending on the parameter R, defined by

$$\mathcal{L}u = \frac{1}{R}\Delta u - (u\cdot\nabla)U - (U\cdot\nabla)u - \nabla p_1, \tag{12}$$

with p_1 given by (7). Note that this linear operator has an integral part, which is the term ∇p_1.

To apply the resolvent technique to prove stability of the stationary flow, it is convenient to have homogeneous initial conditions. Therefore, we transform the problem (11) to a similar problem with homogeneous initial condition by defining

$$v(x,y,t) := u(x,y,t) - e^{-t}f(x,y). \tag{13}$$

Note that v and u have the same behavior as $t\to\infty$. Moreover, v given by (13) satisfies

$$\begin{cases} v_t = \mathcal{L}v - \epsilon\{(v\cdot\nabla)v + e^{-t}(v\cdot\nabla)f + e^{-t}(f\cdot\nabla)v\} - \epsilon\nabla p_2 + F(x,y,t), \\ v(x,0,t) = (0,0), \\ v(x,1,t) = (0,0), \\ v(x,y,t) = v(x+1,y,t), \\ v(x,y,0) = (0,0), \end{cases} \tag{14}$$

where $F(x,y,t) = e^{-t}((\mathcal{L}+I)f - \epsilon e^{-t}(f\cdot\nabla)f)$. Note that F, $F_t \in L_2([0,\infty);L_2(\Omega))$, that is, both $\|F(\cdot,\cdot,t)\|^2$ and $\|F_t(\cdot,\cdot,t)\|^2$ are integrable over $[0,\infty)$.

Our aim is to prove that if ϵ is small enough, then

$$\lim_{t\to\infty} |v(\cdot,t)|_\infty^2 = 0.$$

4. Linear problem

We first consider the general linear problem

$$\begin{cases} v_t = \mathcal{L}v + F(x,y,t), \\ v(x,0,t) = (0,0), \\ v(x,1,t) = (0,0), \\ v(x,y,t) = v(x+1,y,t), \\ v(x,y,0) = (0,0), \end{cases} \tag{15}$$

where $\|F(\cdot,t)\|^2$ and $\|F_t(\cdot,t)\|^2$ integrable over the domain $[0,\infty)$:

$$\int_0^\infty \left(\|F(\cdot,t)\|^2 + \|F_t(\cdot,t)\|^2\right) dt < \infty.$$

In our case of two spatial dimensions, resolvent estimates were found in [1]:

$$\|\widetilde{v}(\cdot,s)\|^2 \le C_1 R^2 \|\widetilde{F}(\cdot,s)\|^2\,, \qquad \mathrm{Re}(s) \ge 0, \tag{16}$$

where $\widetilde{}$ stands for the Laplace transform with respect to t , s is its variable and C_1 is an absolute constant, that is, it does not depend on any of the parameters or functions (readers that are not familiar with the Laplace transform and its basic properties can look at [11], for example).

One can prove, as in [5], Appendix A, that (16) implies

$$\|\widetilde{v}(\cdot,s)\|^2_{\widetilde{H}} \le CR^2 \|\tilde{F}(\cdot,s)\|^2, \tag{17}$$

where C depends on C_1 and on U and its first derivative. Since for our problem U is fixed as the Couette flow, C is an absolute constant as well. From now on, we will use C for any absolute constant, and replace its value as necessary keeping the notation C.

Using Parseval's relation for Laplace transformation, inequality (17) for the transformed functions is translated to the original functions as

$$\int_0^\infty \|v(\cdot,t)\|^2_{\widetilde{H}} dt \le CR^2 \int_0^\infty \|F(\cdot,t)\|^2 dt. \tag{18}$$

Obviously,

$$\int_0^T \|v(\cdot,t)\|^2_{\widetilde{H}} dt \le \int_0^\infty \|v(\cdot,t)\|^2_{\widetilde{H}} dt\,, \qquad \forall\, T \ge 0.$$

Moreover, since the solution of the equation up to time T does not depend on the forcing $F(x,y,t)$ for $t > T$, we have

$$\int_0^T \|v(\cdot,t)\|^2_{\widetilde{H}} dt \le CR^2 \int_0^T \|F(\cdot,t)\|^2 dt\,, \qquad \forall\, T \ge 0. \tag{19}$$

For our argument, we also need similar estimates for v_t. To this end, differentiate equation (15) to get

$$\begin{cases} v_{tt} = \mathcal{L}v_t + F_t(x,y,t), \\ v_t(x,0,t) = (0,0), \\ v_t(x,1,t) = (0,0), \\ v_t(x,y,t) = v_t(x+1,y,t), \\ v_t(x,y,0) = F(x,y,0) =: g(x,y), \end{cases} \tag{20}$$

that is, v_t satisfies an equation of the same type as (15), but with non-homogeneus initial conditions $g(x,y) = F(x,y,0)$. Performing the same type of initialization as before, that is, defining $\varphi := v_t - e^{-t}g$, we get a similar problem for φ, with homogeneus initial conditions and an extra forcing term. Using the estimates for the resolvent, and writing those in terms of v_t, we get

$$\begin{aligned} \int_0^T \|v_t(\cdot,t)\|^2_{\widetilde{H}} dt \ &\leq \| \ F(x,y,0)\|^2_{\widetilde{H}} + CR^2\|(\mathcal{L}+I)F(x,y,0)\|^2 \\ &\quad + CR^2 \int_0^T \|F_t(\cdot,t)\|^2 dt, \qquad \forall T \geq 0. \end{aligned} \tag{21}$$

Combining (19) and (21) gives, for v the solution of (15),

$$\begin{aligned} \int_0^T \left(\|v(\cdot,t)\|^2_{\widetilde{H}} + \|v_t(\cdot,t)\|^2_{\widetilde{H}}\right) dt &\leq \|F(x,y,0)\|^2_{\widetilde{H}} CR^2\|(\mathcal{L}+I)F(x,y,0)\|^2 \\ &\quad + CR^2 \int_0^T \left(\|F(\cdot,t)\|^2 + \|F_t(\cdot,t)\|^2\right) dt, \qquad \forall T \geq 0. \end{aligned} \tag{22}$$

Now, using these estimates for the solution of the linear problem, we can prove a stability result for the nonlinear equation.

5. Stability for the nonlinear problem

The nonlinear problem (14) is

$$\begin{cases} v_t = \mathcal{L}v - \epsilon\{(v\cdot\nabla)v + e^{-t}(v\cdot\nabla)f + e^{-t}(f\cdot\nabla)v\} - \epsilon\nabla p_2 + F(x,y,t), \\ v(x,0,t) = (0,0), \\ v(x,1,t) = (0,0), \\ v(x,y,t) = v(x+1,y,t), \\ v(x,y,0) = (0,0), \end{cases} \tag{23}$$

where $F(x,y,t) = e^{-t}((\mathcal{L}+I)f - \epsilon e^{-t}(f\cdot\nabla)f)$. We prove the following:

Theorem 5.1. *There exists $\epsilon_0 > 0$, $\epsilon_0 = \epsilon_0(R)$, such that if $0 \leq |\epsilon| < \epsilon_0$, then the solution $v(x,y,t)$ of (23) satisfies*

$$\lim_{t\to\infty} |v(\cdot,t)|_\infty = 0.$$

Moreover, $\epsilon_0 = \mathcal{O}(R^{-3})$.

Proof. We consider problem (23) as a linear problem with forcing

$$G(x,y,t) := F(x,y,t) - \epsilon\{(v\cdot\nabla)v + e^{-t}(v\cdot\nabla)f + e^{-t}(f\cdot\nabla)v\} - \epsilon\nabla p_2. \tag{24}$$

Applying inequality (22) with forcing term G gives

$$\int_0^T \Big(\|v(\cdot,t)\|^2_{\widetilde{H}} + \|v_t(\cdot,t)\|^2_{\widetilde{H}}\Big)\,dt \leq \|G(x,y,0)\|^2_{\widetilde{H}} + CR^2\|(\mathcal{L}+I)G(x,y,0)\|^2 + \\ + CR^2\int_0^T \left(\|G(\cdot,t)\|^2 + \|G_t(\cdot,t)\|^2\right)dt, \qquad \forall T\geq 0. \tag{25}$$

From the definition of G, we have

$$\begin{aligned}\int_0^T \Big(\|v(\cdot,t)\|^2_{\widetilde{H}} + \|v_t(\cdot,t)\|^2_{\widetilde{H}}\Big)\,dt &\leq 2\|F(x,y,0)\|^2_{\widetilde{H}} + 2\epsilon^2\|\nabla p_2(x,y,0)\|^2_{\widetilde{H}} \\ &+ CR^2\|(\mathcal{L}_R+\mathcal{I})F(x,y,0)\|^2 + CR^2\|(\mathcal{L}_R+\mathcal{I})p_2(x,y,0)\|^2 \\ + CR^2\int_0^T &\left(\|F - \epsilon\{(v\cdot\nabla)v + e^{-t}(v\cdot\nabla)f + e^{-t}(f\cdot\nabla)v\} - \epsilon\nabla p_2\|^2\right)dt \\ + CR^2\int_0^T\Big(\|(F - \epsilon\{(v\cdot\nabla)v &+ e^{-t}(v\cdot\nabla)f + e^{-t}(f\cdot\nabla)v\} - \epsilon\nabla p_2)_t\|^2\Big)\,dt.\end{aligned} \tag{26}$$

Since p_2 is given by (8), one can prove that

$$\|\nabla p_2\| \leq \|(u\cdot\nabla)u\|\,; \qquad \|(\nabla p_2)_t\| \leq \|((u\cdot\nabla)u)_t\|.$$

Thus, using (13), we can estimate ∇p_2 by f and v. Moreover,

$$\|\nabla p_2(\cdot,\cdot,0)\|^2 \leq \|(u\cdot\nabla)u(\cdot,\cdot,0)\|^2 = \|(f\cdot\nabla)f\|^2,$$

and since $\|f\|^2_{H^6} = 1$, inequality (26) gives

$$\begin{aligned}\int_0^T \Big(\|v(\cdot,t)\|^2_{\widetilde{H}} + \|v_t(\cdot,t)\|^2_{\widetilde{H}}\Big)\,dt &\leq \|F(x,y,0)\|^2_{\widetilde{H}} + CR^2\|(\mathcal{L}+I)F(x,y,0)\|^2 \\ &+ CR^2\int_0^\infty \left(\|F\|^2 + \|F_t\|^2\right)dt \\ &+ CR^2\epsilon^2\int_0^T \left(\|(v\cdot\nabla)v\|^2 + \|(v_t\cdot\nabla)v\|^2 + \|(v\cdot\nabla)v_t\|^2\right)dt \\ + CR^2\epsilon^2\int_0^T\Big(\|e^{-t}(v\cdot\nabla)f\|^2 &+ \|e^{-t}(f\cdot\nabla)v\|^2 + \|e^{-t}(v_t\cdot\nabla)f\|^2 + \|e^{-t}(f\cdot\nabla)v_t\|^2\Big)\,dt.\end{aligned}$$

Since

$$F(x,y,t) = e^{-t}((\mathcal{L}+I)f - \epsilon e^{-t}(f\cdot\nabla)f),$$

we have $F(x,y,0) = (\mathcal{L}+I)f - \epsilon(f\cdot\nabla)f := \mathcal{P}f$. With this notation, the inequality above is

$$\begin{aligned}\int_0^T \left(\|v(\cdot,t)\|_{\widetilde{H}}^2 + \|v_t(\cdot,t)\|_{\widetilde{H}}^2\right) dt &\le \|\mathcal{P}f\|_{\widetilde{H}}^2 + CR^2\|(\mathcal{L}+I)\mathcal{P}f\|^2 + CR^2\|(\mathcal{L}+I)f\|^2 \\ &+CR^2\epsilon^2\|(f\cdot\nabla)f\|^2 + CR^2\epsilon^2\int_0^T \left(\|(v\cdot\nabla)v\|^2 + \|(v_t\cdot\nabla)v\|^2 + \|(v\cdot\nabla)v_t\|^2\right) dt \qquad (27)\\ &+CR^2\epsilon^2\int_0^T\left(\|e^{-t}(v\cdot\nabla)f\|^2 + \|e^{-t}(f\cdot\nabla)v\|^2 + \|e^{-t}(v_t\cdot\nabla)f\|^2 + \|e^{-t}(f\cdot\nabla)v_t\|^2\right) dt.\end{aligned}$$

It is not difficult to check that all the terms depending on f of the right hand side of inequality (27) can be bounded by $C\|f\|_{H^6}$. Therefore, since $\|f\|_{H^6}^2 = 1$, we replace all these terms by an absolute constant and write inequality (27) as

$$\begin{aligned}\int_0^T \left(\|v\|_{\widetilde{H}}^2 + \|v_t\|_{\widetilde{H}}^2\right) dt &\le CR^2 + CR^2\epsilon^2\int_0^T \|(v\cdot\nabla)v\|^2 dt \\ &+ CR^2\epsilon^2\int_0^T \left(\|(v_t\cdot\nabla)v\|^2 + \|(v\cdot\nabla)v_t\|^2\right) dt \\ &+ CR^2\epsilon^2\int_0^T \left(\|e^{-t}(v\cdot\nabla)f\|^2 + \|e^{-t}(f\cdot\nabla)v\|^2\right) dt \\ &+ CR^2\epsilon^2\int_0^T \left(\|e^{-t}(v_t\cdot\nabla)f\|^2 + \|e^{-t}(f\cdot\nabla)v_t\|^2\right) dt. \qquad (28)\end{aligned}$$

From now on, we fix the constant C. To finish the proof, we use the following Lemma, which is proved in appendix A:

Lemma 5.2. *There exists $\epsilon_0 > 0$, $\epsilon_0 = \mathcal{O}(R^{-3})$, such that if $0 \le \epsilon < \epsilon_0$ then*

$$\int_0^T \left(\|v(\cdot,t)\|_{\widetilde{H}}^2 + \|v_t(\cdot,t)\|_{\widetilde{H}}^2\right) dt < 2CR^2, \quad \forall T \ge 0. \qquad (29)$$

Now, using (5) and a simple one dimensional Sobolev inequality (see appendix B), we have

$$\begin{aligned}\max_{a\le t\le b} |v(\cdot,t)|_\infty^2 &\le \widetilde{C}R \max_{a\le t\le b}\|v(\cdot,t)\|_{\widetilde{H}}^2 \\ &\le \widetilde{C}R\left(1+\frac{1}{b-a}\right)\int_a^b \|v(\cdot,t)\|_{\widetilde{H}}^2 dt + \int_a^b \|v_t(\cdot,t)\|_{\widetilde{H}}^2 dt.\end{aligned}$$

This implies

$$\sup_{a\le t} |v(\cdot,t)|_\infty^2 \le \widetilde{C}R\int_a^\infty \left(\|v(\cdot,t)\|_{\widetilde{H}}^2 + \|v_t(\cdot,t)\|_{\widetilde{H}}^2\right) dt. \qquad (30)$$

Note that in view of Lemma 5.2, the right hand side of inequality (30) is finite. Letting $a \to \infty$ in (30), we have that $\lim_{t\to\infty} |v(\cdot,t)|_\infty^2 = 0$, which proves the theorem.

Appendix

A. Proof of Lemma 5.2

First, note that for $T > 0$ small enough, inequality (29) obviously holds. Now, suppose it does not hold for all $T \geq 0$, that is, there exists $T_0 > 0$ such that

$$\int_0^{T_0} \left(\|v(\cdot,t)\|_{\widetilde{H}}^2 + \|v_t(\cdot,t)\|_{\widetilde{H}}^2 \right) dt = 2CR^2. \tag{31}$$

Using (28), we have:

$$\begin{aligned} 2CR^2 = \int_0^{T_0} \left(\|v\|_{\widetilde{H}}^2 + \|v_t\|_{\widetilde{H}}^2 \right) dt \leq \\ \leq CR^2 + CR^2\epsilon^2 \int_0^{T_0} \left(\|(v\cdot\nabla)v\|^2 + \|(v_t\cdot\nabla)v\|^2 + \|(v\cdot\nabla)v_t\|^2 \right) dt \\ + \int_0^{T_0} \left(\|e^{-t}(v\cdot\nabla)f\|^2 + \|e^{-t}(f\cdot\nabla)v\|^2 + \|e^{-t}(v_t\cdot\nabla)f\|^2 + \|e^{-t}(f\cdot\nabla)v_t\|^2 \right) dt. \end{aligned} \tag{32}$$

We now estimate the integrands on the right hand side of inequality (32) by the integral on its left hand side. To this end, we will use the inequalities (5) and

$$\max_{0\leq t\leq T_0} \|v(\cdot,t)\|_{\widetilde{H}}^2 \leq \int_0^{T_0} \left(\|v(\cdot,t)\|_{\widetilde{H}}^2 + \|v_t(\cdot,t)\|_{\widetilde{H}}^2 \right) dt = 2CR^2. \tag{33}$$

Since $\|v\|_{\widetilde{H}}^2 = \|v\|^2 + \frac{1}{R}\|Dv\|^2 + \frac{1}{R^2}\|v_{xy}\|^2$, we have $\|Dv\|^2 \leq R\|v\|_{\widetilde{H}}^2$. Therefore, for each $0 \leq t \leq T_0$:

$$\begin{aligned} \|\{(v\cdot\nabla)v\}(\cdot,t)\|^2 &\leq |v(\cdot,t)|_\infty^2 \|Dv(\cdot,t)\|^2 \\ &\leq \left(\widetilde{C}R\|v(\cdot,t)\|_{\widetilde{H}}^2\right)\left(R\|v(\cdot,t)\|_{\widetilde{H}}^2\right) \\ &\leq 2\widetilde{C}CR^4\|v(\cdot,t)\|_{\widetilde{H}}^2, \end{aligned} \tag{34}$$

$$\begin{aligned} \|\{(v_t\cdot\nabla)v\}(\cdot,t)\|^2 &\leq |v_t(\cdot,t)|_\infty^2 \|Dv(\cdot,t)\|^2 \\ &\leq \left(\widetilde{C}R\|v_t(\cdot,t)\|_{\widetilde{H}}^2\right)\left(R\|v(\cdot,t)\|_{\widetilde{H}}^2\right) \\ &\leq 2\widetilde{C}CR^4\|v_t(\cdot,t)\|_{\widetilde{H}}^2, \end{aligned} \tag{35}$$

$$\begin{aligned} \|\{(v\cdot\nabla)v_t\}(\cdot,t)\|^2 &\leq |v(\cdot,t)|_\infty^2 \|Dv_t(\cdot,t)\|^2 \\ &\leq \left(\widetilde{C}R\|v(\cdot,t)\|_{\widetilde{H}}^2\right)\left(R\|v_t(\cdot,t)\|_{\widetilde{H}}^2\right) \\ &\leq 2\widetilde{C}CR^4\|v_t(\cdot,t)\|_{\widetilde{H}}^2, \end{aligned} \tag{36}$$

$$\|e^{-t}\{(v\cdot\nabla)f\}(\cdot,t)\|^2 \leq e^{-2t}|v(\cdot,t)|_\infty^2\|Df\|^2 \leq \widetilde{C}R\|v(\cdot,t)\|_{\widetilde{H}}^2, \tag{37}$$

$$\|e^{-t}\left\{(f\cdot\nabla)v\right\}(\cdot,t)\|^2 \le e^{-2t}|f|_\infty^2\|Dv(\cdot,t)\|^2 \le R\|v(\cdot,t)\|_{\widetilde{H}}^2, \tag{38}$$

$$\|e^{-t}\left\{(v_t\cdot\nabla)f\right\}(\cdot,t)\|^2 \le e^{-2t}|v_t(\cdot,t)|_\infty^2\|Df\|^2 \le \widetilde{C}R\|v_t(\cdot,t)\|_{\widetilde{H}}^2, \tag{39}$$

$$\|e^{-t}\left\{(f\cdot\nabla)v_t\right\}(\cdot,t)\|^2 \le e^{-2t}|f|_\infty^2\|Dv_t(\cdot,t)\|^2 \le R\|v_t(\cdot,t)\|_{\widetilde{H}}^2. \tag{40}$$

Applying (34), (35), (36), (37), (38), (39), (40) to (32) gives

$$\begin{aligned} 2CR^2 &\le CR^2 + CR^2\epsilon^2\left\{6\widetilde{C}CR^4\int_0^{T_0}\left(\|v(\cdot,t)\|_{\widetilde{H}}^2 + \|v_t(\cdot,t)\|_{\widetilde{H}}^2\right)dt\right\} \\ &= CR^2 + CR^2\epsilon^2\left\{12\widetilde{C}C^2R^6\right\}. \end{aligned} \tag{41}$$

This implies

$$1 \le 12\widetilde{C}C^2R^6\epsilon^2, \tag{42}$$

which is equivalent to

$$\epsilon \ge \frac{1}{CR^3\sqrt{12\widetilde{C}}} = \frac{1}{KR^3}, \tag{43}$$

where $K := C\sqrt{12\widetilde{C}}$. Therefore, if $\epsilon < \dfrac{1}{KR^3}$, equality (31) never holds. This proves the Lemma. ☑

B. Proof of the Sobolev type inequalities

We state and prove the Sobolev type inequalities used here. The euclidian inner product and norm in $\mathbb{R}^n$ are denoted by $(\cdot,\cdot)$ and $|\cdot|$, respectively.

Lemma B.1. *Let $f(t):[a,b]\to\mathbb{R}^n$ be a C^1 function. Then,*

$$\max_{a\le t\le b}|f(t)|^2 \le \left(1+\frac{1}{b-a}\right)\int_a^b|f(t)|^2dt + \int_a^b|f_t(t)|^2dt. \tag{44}$$

Moreover, if there exists $t_0\in[a,b]$ such that $f(t_0)=0$, then

$$\max_{a\le t\le b}|f(t)|^2 \le \int_a^b(|f(t)|^2+|f_t(t)|^2)dt. \tag{45}$$

Proof. Let t_1, $t_2\in[a,b]$ such that

$$|f(t_1)|^2 = \min_{a\le t\le b}|f(t)|^2 \qquad \text{y} \qquad |f(t_2)|^2 = \max_{a\le t\le b}|f(t)|^2.$$

Then, from

$$\frac{d}{dt}|f(t)|^2 = 2\left(f(t),f'(t)\right) \le 2|f(t)||f'(t)| \le |f(t)|^2+|f'(t)|^2 \tag{46}$$

we have

$$\begin{aligned}|f(t_2)|^2 - |f(t_1)|^2 &= \int_{t_1}^{t_2} \frac{d}{dt}|f(t)|^2 dt \ \leq \ \int_{t_1}^{t_2} \left(|f(t)|^2 + |f'(t)|^2\right) dt \\ &\leq \int_a^b \left(|f(t)|^2 + |f'(t)|^2\right) dt,\end{aligned}$$

and then

$$\max_{a\leq t\leq b} |f(t)|^2 = |f(t_2)|^2 \leq |f(t_1)|^2 + \int_a^b \left(|f(t)|^2 + |f'(t)|^2\right) dt. \tag{47}$$

Since

$$|f(t_1)|^2 = \min_{a\leq t\leq b} |f(t)|^2 \leq \frac{1}{b-a} \int_a^b |f(t)|^2 dt,$$

(47) implies the desired inequality

$$\max_{a\leq t\leq b} |f(t)|^2 \leq \left(1 + \frac{1}{b-a}\right) \int_a^b |f(t)|^2 dt + \int_a^b |f_t(t)|^2 dt. \tag{48}$$

If there exists $t_0 \in [a,b]$ such that $f(t_0) = 0$, then $\min\limits_{a\leq t\leq b} |f(t)|^2 = 0$ and the proof above gives

$$\max_{a\leq t\leq b} |f(t)|^2 \leq \int_a^b (|f(t)|^2 + |f_t(t)|^2) dt. \tag{49}$$

This finishes the proof. ☑

For completeness, we state the following Lemma which is a slight variation of Lemma B.1. It will be used to prove inequality (4).

Lemma B.2. *Let $f : [0,1] \to \mathbb{R}^n$ be a C^1 function. Then,*

$$|f|_\infty^2 \leq \|f\|^2 + 2\|f\|\|f'\|. \tag{50}$$

Proof. The proof is basically the same as the proof of Lemma B.1: to get the result, just keep the inner product in (46) and use the Cauchy-Schwartz inequality. ☑

We now prove inequality (4). We prove a general inequality valid for functions of two variables defined in a strip. For the general case of more dimensions, we refer to [2], Appendix 3. The proof given here follows the proof of this general case.

Lemma B.3. *Let $f : \mathbb{R} \times [0,1] \to \mathbb{R}^2$ be a C^∞ function, 1-periodic in x. For any ϵ, $0 < \epsilon \leq 1$,*

$$|f|_\infty^2 \leq (2 + 2\pi) \left(\frac{1}{\epsilon^2}\|f\|^2 + \|f_x\|^2 + \|f_y\|^2 + \epsilon^2\|f_{xy}\|^2\right). \tag{51}$$

Proof. We will use the representation of f as a Fourier sum in the periodic direction x and the Cauchy-Schwarz inequality

$$\begin{aligned} |f(x,y)|^2 &\leq \left(\sum_{k\in\mathbb{Z}} |\widehat{f}(k,y)|\right)^2 = \left(\sum_{k\in\mathbb{Z}} |\widehat{f}(k,y)|(1+\epsilon^2k^2)^{\frac{1}{2}}(1+\epsilon^2k^2)^{-\frac{1}{2}}\right)^2 \\ &\leq \left(\sum_{k\in\mathbb{Z}} |\widehat{f}(k,y)|^2(1+\epsilon^2k^2)\right)\left(\sum_{k\in\mathbb{Z}}(1+\epsilon^2k^2)^{-1}\right). \end{aligned} \tag{52}$$

Now, estimate each of the factors in the product above. First,

$$\sum_{k\in\mathbb{Z}}(1+\epsilon^2k^2)^{-1} = 1 + 2\sum_{k=1}^{\infty}\frac{1}{1+\epsilon^2k^2} \leq 1 + 2\int_0^{\infty}\frac{1}{1+\epsilon^2k^2}dk.$$

Since

$$\int_0^{\infty}\frac{1}{1+\epsilon^2k^2}dk = \frac{1}{\epsilon}\int_0^{\infty}\frac{1}{1+\xi^2}d\xi = \frac{\pi}{2\epsilon},$$

we conclude that

$$\sum_{k\in\mathbb{Z}}(1+\epsilon^2k^2)^{-1} \leq 1 + \frac{\pi}{\epsilon} \leq \frac{1+\pi}{\epsilon}. \tag{53}$$

To estimate the other factor, we use the inequality (50) to write

$$\begin{aligned} |\widehat{f}(k,y)|^2 &\leq \|\widehat{f}(k,\cdot)\|^2 + 2\|\widehat{f}(k,\cdot)\|\|\widehat{f}_y(k,\cdot)\| \\ &\leq \|\widehat{f}(k,\cdot)\|^2 + \frac{1}{\epsilon}\|\widehat{f}(k,\cdot)\|^2 + \epsilon\|\widehat{f}_y(k,\cdot)\|^2, \end{aligned}$$

which implies

$$\sum_{k\in\mathbb{Z}} |\widehat{f}(k,y)|^2 \leq \|f\|^2 + \frac{1}{\epsilon}\|f\|^2 + \epsilon\|f_y\|^2 \leq \frac{2}{\epsilon}\|f\|^2 + \epsilon\|f_y\|^2. \tag{54}$$

Moreover,

$$\begin{aligned} k^2|\widehat{f}(k,y)|^2 &\leq k^2\|\widehat{f}(k,\cdot)\|^2 + 2\left(|k|\|\widehat{f}(k,\cdot)\|\right)\left(|k|\|\widehat{f}_y(k,\cdot)\|\right) \\ &\leq k^2\|\widehat{f}(k,\cdot)\|^2 + \frac{1}{\epsilon}k^2\|\widehat{f}(k,\cdot)\|^2 + \epsilon k^2\|\widehat{f}_y(k,\cdot)\|^2 \\ &\leq \frac{2}{\epsilon}k^2\|\widehat{f}(k,\cdot)\|^2 + \epsilon k^2\|\widehat{f}_y(k,\cdot)\|^2. \end{aligned}$$

Summing this inequality over k, we obtain

$$\sum_{k\in\mathbb{Z}} k^2|\widehat{f}(k,y)|^2 \leq \frac{2}{\epsilon}\|f_x\|^2 + \epsilon\|f_{xy}\|^2,$$

which implies

$$\sum_{k\in\mathbb{Z}} \epsilon^2k^2|\widehat{f}(k,y)|^2 \leq 2\epsilon\|f_x\|^2 + \epsilon^3\|f_{xy}\|^2. \tag{55}$$

Applying (53), (54), (55) to (52), we get

$$|f(x,y)|^2 \leq \left(\frac{1+\pi}{\epsilon}\right)\left(\frac{2}{\epsilon}\|f\|^2 + \epsilon\|f_y\|^2 + 2\epsilon\|f_x\|^2 + \epsilon^3\|f_{xy}\|^2\right),$$

which implies the desired conclusion

$$|f|_\infty^2 \leq (2+2\pi)\left(\frac{1}{\epsilon^2}\|f\|^2 + \|f_x\|^2 + \|f_y\|^2 + \epsilon^2\|f_{xy}\|^2\right). \tag{56}$$

Inequality (4) is a special case of (51). Indeed, since

$$\|u\|_{\widetilde{H}}^2 = \|u\|^2 + \frac{1}{R}\|Du\|^2 + \frac{1}{R^2}\|u_{xy}\|^2,$$

we have

$$R\|u\|_{\widetilde{H}}^2 = R\|u\|^2 + \|Du\|^2 + \frac{1}{R}\|u_{xy}\|^2,$$

and (51) with $\epsilon^2 = \dfrac{1}{R}$ implies

$$|u|_\infty^2 \leq (2+2\pi)R\|u\|_{\widetilde{H}}^2. \qquad ☑$$

Acknowledgment. Supported by a post-doctoral fellowship FAPESP/Brazil: 02/13270-1.

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